International Journal of Theoretical Physics, Vol. 39, No. 9, 2000

Entropy of the Dirac Field in Schwarzschild–de Sitter Space-Time via the Membrane Model

Changjun Gao[1] and Wenbiao Liu[2]

Received April 6, 2000

There are two event horizons in Schwarzschild–de Sitter space-time, a black hole horizon and a cosmological horizon. They have different temperatures. The radiation between them is of course not in thermal equilibrium. According to the membrane model suggested by us, the two horizons can be thought of as two independent thermodynamic systems in equilibrium. Their Dirac field entropies are calculated via a membrane model. The result shows that the entropy of the Dirac field is proportional to the sum of the areas of the two event horizons. If we choose the same cutoff as that of Klein–Gordon field, the entropy of the Dirac field is $3\frac{1}{2}$ times that of Klein–Gordon field. This agrees with previous results.

1. INTRODUCTION

The entropy of a black hole is proportional to the area of its event horizon [3, 5, 8]. The origin of black hole entropy remains a fascinating problem. 't Hooft [2] presented a "brick wall" model that can be used to compute the black hole entropy. Various work followed that calculated black hole entropy via this model [4, 6, 10].

However, the entropy of the Dirac field in Schwarzschild–de Sitter (SD) space-time has not been calculated via the brick wall model because, unlike ordinary stationary space-time, there are two event horizons in SD space-time, which have different temperatures [9]. The radiation between them is of course not in equilibrium, and thus we cannot take the brick wall model, which is based on equilibrium statistical physics. In the membrane model

[1] National Astronomical Observation, Chinese Academy of Sciences, Beijing, 100012, P.R. China; Shanghi Astronomical Observatory, Chinese Academy of Sciences, Shanghi, 200030, P.R. China.
[2] Department of Physics, Beijing Normal University, Beijing, 100875, China.

0020-7748/00/0900-2221$18.00/0

the entropy of the black hole comes from the entropy of a thin layer of radiation [1]. It does not matter whether the two event horizons have the same temperature or not and whether their distance is large enough or not. Therefore we can regard the two horizons as two independent thermodynamic systems in equilibrium and calculate the entropy of the Dirac field via the membrane model. Our result agrees with previous results [7].

2. DIRAC EQUATION IN SCHWARZSCHILD–DE SITTER SPACE-TIME

The Dirac equation in curved space-time is given by

$$\begin{aligned}
(D + \varepsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 &= \frac{i}{\sqrt{2}}\mu_0 G_1 \\
(D + \varepsilon - \rho)F_1 + (\bar{\delta} + \pi - \alpha)F_2 &= \frac{i}{\sqrt{2}}\mu_0 G_1 \\
(D + \varepsilon^* - \rho^*)G_2 + (\delta + \pi^* - \alpha^*)G_1 &= \frac{i}{\sqrt{2}}\mu_0 F_1 \\
(\Delta' + \mu^* - \gamma^*)G_1 + (\bar{\delta} + \beta^* - \tau^*)G_2 &= \frac{i}{\sqrt{2}}\mu_0 F_1
\end{aligned} \tag{1}$$

where μ_0 is the mass of particle, F_1, F_2, G_1, and G_2 are four components of the wave function, D, Δ', δ, and $\bar{\delta}$ are ordinary differential operators, and α, β, γ, δ, etc., are spin coefficients. Their relation to the null tetrad is

$$\begin{aligned}
\alpha &= \tfrac{1}{2}(l_{\mu;\nu}n\mu\bar{m}\nu - m_{\mu;\nu}\bar{m}\mu\bar{m}\nu) \\
\beta &= \tfrac{1}{2}(l_{\mu;\nu}n\mu m\nu - m_{\mu;\nu}\bar{m}\mu m\nu) \\
\rho &= l_{\mu;\nu}m\mu\bar{m}\nu \\
\pi &= -n_{\mu;\nu}\bar{m}\mu l\nu
\end{aligned}$$

with

$$\begin{aligned}
D &= l\mu\partial\mu \\
\Delta' &= n\mu\partial\mu \\
\delta &= m\mu\partial\mu \\
\bar{\delta} &= \bar{m}\partial\mu
\end{aligned} \tag{2}$$

and

$$\gamma = \tfrac{1}{2}(l_{\mu;\nu} n\mu n\nu - m_{\mu;\nu} \overline{m}\mu n\nu)$$
$$\epsilon = \tfrac{1}{2}(l_{\mu;\nu} n\mu l\nu - m_{\mu;\nu} \overline{m}\mu l\nu)$$
$$\mu = -n_{\mu;\nu} \overline{m}\mu m\nu$$
$$\tau = -l_{\mu;\nu}\, m\mu n\nu$$

The null tetrad satisfies the following equations:

$$l\mu n\mu = -m\mu\overline{m}\mu = 1$$
$$l\mu l\mu = n\mu n\mu = m\mu m\mu = \overline{m}\mu\overline{m}\mu = 0$$
$$l\mu m\mu = l\mu\overline{m}\mu = n\mu m\mu = n\mu\overline{m}\mu = 0$$
$$g_{\mu\nu} = l\mu n\nu + n\mu l\nu + m\mu\overline{m}\nu - \overline{m}\mu m\nu \tag{3}$$

The line element of the SD space-time is

$$ds^2 = (1 - 2M/r - \tfrac{1}{3}\lambda r^2)\, dt^2 - (1 - 2M/r - \tfrac{1}{3}\lambda r^2)^{-1}\, dr^2 - r^2\, d\Omega^2 \tag{4}$$

We choose the null tetrad as follows:

$$l\mu = \frac{1}{\Delta}\,(r^2, \Delta, 0,0)$$
$$n\mu = \frac{1}{2r^2}\,(r^2, -\Delta, 0, 0)$$
$$m\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, \frac{i}{\sin\theta}\right)$$
$$\overline{m}\mu = \frac{1}{\sqrt{2}r}\left(0, 0, 1, -\frac{i}{\sin\theta}\right) \tag{5}$$

where $\Delta = r^2 - 2Mr - \tfrac{1}{3}\lambda r^4$. In view of the symmetry of space-time, let the four components of the wave function be

$$F_1 = e^{-iEt}\, e^{im\phi}\, r^{-1} f_1(r, \theta)$$
$$F_2 = e^{-iEt}\, e^{im\phi}\, f_2(r, \theta)$$
$$G_1 = e^{-iEt}\, e^{im\phi}\, g_1(r, \theta)$$
$$G_2 = e^{-iEt}\, e^{im\phi}\, r^{-1} g_2(r, \theta) \tag{6}$$

Then Eqs. (1) become

$$\begin{aligned} D_0 f_1 + \frac{1}{\sqrt{2}} L^+_{1/2} f_2 &= \frac{1}{\sqrt{2}} i\mu_0 r g_1 \\ \Delta D^+_{1/2} f_2 - \sqrt{2} L_{1/2} f_1 &= -\sqrt{2} i\mu_0 r g_2 \\ D_0 g_2 - \frac{1}{\sqrt{2}} L_{1/2} g_1 &= \frac{1}{\sqrt{2}} i\mu_0 r f_2 \\ \Delta D^+_{1/2} g_1 + \sqrt{2} L^+_{1/2} g_2 &= -\sqrt{2} i\mu_0 r f_1 \end{aligned} \tag{7}$$

where

$$\begin{aligned} D_n &= \partial_r - iE\frac{r^2}{\Delta} + 2n\frac{r - M - \frac{2}{3}\lambda r^3}{\Delta} \\ D^+_n &= \partial_r + iE\frac{r^2}{\Delta} + 2n\frac{r - M - \frac{2}{3}\lambda r^3}{\Delta} \\ L^+_n &= \partial\theta + \frac{m}{\sin\theta} + n \operatorname{ctg}\theta \\ L_n &= \partial\theta - \frac{m}{\sin\theta} + n \operatorname{ctg}\theta \end{aligned} \tag{8}$$

To get the decoupling equations with regard to $R_{\pm 1/2}$, and $\Theta_{\pm 1/2}$, we suppose that

$$\begin{aligned} f_1(r, \theta) &= R_{-1/2}(r)\Theta_{-1/2}(\theta) \\ f_2(r, \theta) &= R_{+1/2}(r)\Theta_{+1/2}(\theta) \\ g_1(r, \theta) &= R_{+1/2}(r)\Theta_{-1/2}(\theta) \\ g_2(r, \theta) &= R_{-1/2}(r)\Theta_{+1/2}(\theta) \end{aligned} \tag{9}$$

For simplicity we put μ_0 zero, so we get

$$\begin{aligned} \Delta D^+_{1/2} D_0 R_{-1/2} &= l(l + 1)R_{-1/2} \\ D_0 \Delta D^+_{1/2} R_{+1/2} &= l(l + 1)R_{+1/2} \\ L_{1/2} L^+_{1/2} \Theta_{+1/2} &= -l(l + 1)\Theta_{+1/2} \\ L^+_{1/2} L_{1/2} \Theta_{-1/2} &= -l(l + 1)\Theta_{-1/2} \end{aligned} \tag{10}$$

Expandings Eqs. (10), we obtain the radial equations as follows:

$$-\Delta \frac{d^2R_{-1/2}}{dr^2} - \left(r - M - \frac{2}{3}\lambda r^3\right)\frac{dR_{-1/2}}{dr} - \frac{E^2r^4}{\Delta}R_{-1/2}$$

$$= \left[-2iEr + \frac{iEr\left(r - M - \frac{2}{3}\lambda r^3\right)}{\Delta} - l(l+1)\right]R_{-1/2} \tag{11}$$

$$-\Delta \frac{d^2R_{+1/2}}{dr^2} - 3\left(r - M - \frac{2}{3}\lambda r^3\right)\frac{dR_{+1/2}}{dr} - \left(\frac{E^2r^4}{\Delta} + 2iEr\right)R_{+1/2}$$

$$= \left[(1 - 2\lambda r^2) - \frac{iEr\left(r - M - \frac{2}{3}\lambda r^3\right)}{\Delta} - l(l+1)\right]R_{+1/2} \tag{12}$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} - \frac{m^2}{\sin^2\theta} + l(l+1)\right]\Theta_{\pm 1/2}$$

$$= -\left[\frac{1}{4}\operatorname{ctg}^2\theta \mp \frac{m\cos\theta}{\sin^2\theta} - \frac{1}{2\sin^2\theta}\right]\Theta_{\pm 1/2} \tag{13}$$

In this paper only the radial equations (11) and (12) are considered.

3. THE ENTROPY OF THE DIRAC FIELD IN THE BACKGROUND OF SD SPACE-TIME

There are four components in the wave function of the Dirac field. We can calculate the entropy of each component and then sum them.

The line element of SD space-time is

$$ds^2 = (1 - 2M/r - \tfrac{1}{3}\lambda r^2)\,dt^2 - (1 - 2M/r - \tfrac{1}{3}\lambda r^2)^{-1}\,dr^2 - r^2\,d\Omega^2 \tag{14}$$

The equation of the event horizon can be easily obtained from the above line element,

$$r^3 + 6M/\lambda - 3r/\lambda = 0 \tag{15}$$

Let r_+, r_c, and r_0 be respectively, the black hole event horizon, the cosmological event horizon, and a meaningless root; then we can write Eq. (15) as

$$(r - r_+)(r - r_-)(r - r_0) = 0 \tag{16}$$

The temperatures of the two event horizons are

$$\beta_{+}^{-1} = T_{+} = \left|\frac{M - \frac{1}{3}\lambda r_{+}^{3}}{2\pi r_{+}^{2}}\right| \tag{17}$$

$$\beta_{c}^{-1} = T_{c} = \left|\frac{M - \frac{1}{3}\lambda r_{c}^{3}}{2\pi r_{c}^{2}}\right| \tag{18}$$

First, let us consider the case of the black hole event horizon.

3.1. Entropy of $f_1(r, \theta)$ Component

The boundary condition is $R_{\pm} = 0$ when $r < r_{+} + h_1$ or $r > r_{+} + 2h_1$, where $h_1 \gg r_{+}$. We use the WKB approximation, i.e., let $R_{-1/2} = e^{iS_1(r)}$, and put it into (11); the wave number is

$$k_1^2(r, l, \omega) = \left(\frac{dR_{-1/2}}{dr}\right)^2 = \left(1 - \frac{2M}{r} - \frac{1}{3}\lambda r^2\right)^{-1} \times \left[\left(1 - \frac{2M}{r} - \frac{1}{3}\lambda r^2\right)^{-1} E^2 - \frac{l(l+1)}{r^2}\right] \tag{19}$$

Considering semiclassical quantum theory, we have for the constraint imposed on the wave number k

$$n_1\pi = \int_{r_{+}+h_1}^{r_{+}+2h_1} dr\, k_1(r, l, \omega) \tag{20}$$

The free energy is given by statistical physics,

$$\beta_{+} f_1 = \sum_{n_1, l, m} \ln(1 + e^{-\beta_{+} E}) \tag{21}$$

The distribution of the state density is regarded as being continuous. Then we get

$$\beta_{+} f_1 = \int dl\, (2l + 1) \int dn_1\, \ln(1 + e^{-\beta_{+} E}) \tag{22}$$

Integrating by parts and substituting Eq. (19) for Eq. (20), we find that Eq. (22) becomes

$$\beta_{+} f_1 = -\frac{2\beta_{+}}{3\pi} \int_{r=+h_1}^{r_{+}+2h_1} \left(1 - \frac{2M}{r} - \frac{1}{3}\lambda r^2\right)^{-2} r^2 \int_0^{\infty} dE\, (\mathrm{e}^{\beta_{+} E} + 1)^{-1} E^3 \tag{23}$$

When integrating over l the part under the radical sign that is not negative should be considered. Using the median theorem in calculus and considering $h_1 \gg r_+$, we get

$$f_1 \approx -\frac{7r_+^4\pi^3}{20(r_+ - r_c)^2(r_+ - r_0)^2\lambda^2\beta_+^4}\frac{h_1}{\eta_1^2} \tag{24}$$

where $h_1 < \eta_1 < 2h_1$.

The entropy is given by ensemble theory,

$$S = \beta^2\frac{\partial F}{\partial\beta} \tag{25}$$

Therefore

$$S_{11} = \frac{7\pi^3 r_+^4}{5\beta_+^3(r_+ - r_c)^2(r_+ - r_0)^2\lambda^2}\frac{h_1}{\eta_1^2} \tag{26}$$

3.2. Entropy of $f_2(r, \theta)$ Component

Similar to the process of computing the entropy of the $f_1(r, \theta)$ component, we get the entropy of the $f_2(r, \theta)$ component:

$$S_{12} = S_{11} \tag{27}$$

3.3. Entropy of the Black Hole Event Horizon

Since the radial components of $g_1(r, \theta)$ and $g_2(r, \theta)$ are the same as those of $f_1(r, \theta)$ and $f_2(r, \theta)$, respectively, their entropies are also same. Thus the total entropy is

$$\begin{aligned} S_+ &= S_{11} + S_{12} + S_{13} + S_{14} = 4S_{11} \\ &= \frac{28\pi^3 r_+^4}{5\beta_+^3(r_+ - r_c)^2(r_+ - r_0)^2\lambda^2}\frac{h_1}{\eta_1^2} \end{aligned} \tag{28}$$

When we choose

$$\frac{h_1}{\eta_1^2} = 90\beta_+ \tag{29}$$

we find that the entropy of the Dirac field is $3\frac{1}{2}$ times that of the Klein–Gordon field, i.e.,

$$S_+ = \frac{7}{2}\frac{1}{4}A_+ \tag{30}$$

The proof is given in Section 4.

Now we give the entropy of the cosmological event horizon.

The boundary condition is $R_{\pm} = 0$ when $r < r_c - 2h_2$ or $r > r_c - h_2$, where $h_2 \gg r_c$. Entirely as in the case of a black hole, the entropy of the cosmological event horizon can be derived, and is

$$S_c = \frac{28\pi^3 r_+^4}{5\beta_c^3 (r_+ - r_c)^2 (r_c - r_0)^2 \lambda^2} \frac{h_2}{\eta_2^2} \tag{31}$$

When we choose

$$\frac{h_2}{\eta_2^2} = 90\beta_c \tag{32}$$

we find the entropy of the Dirac field to be $3\frac{1}{2}$ times that of the Klein–Gordon field, i.e.,

$$S_c = \frac{7}{2}\frac{1}{4} A_c \tag{33}$$

The proof is given in Section 4.

Now we obtain the total entropy of the Dirac field in the background of SD space-time as,

$$S = S_+ + S_c = \frac{7}{2}\frac{1}{4} A \tag{34}$$

where A is total area of the two horizons.

4. PROOF OF EQUATION (30)

Since the proof of Eq. (33) is similar to that of Eq. (30), we give the proof of Eq. (30) only. Putting Eqs. (17) and (29) into (28), we obtain

$$S_+ = 4\pi r_+^2 \frac{(r_+^3 - (3/\lambda)M)^2}{r_+^2 (r_+ - r_c)^2 (r_+ - r_0)^2} \frac{7}{2} \tag{35}$$

The relations between the roots and the coefficients in Eq. (16) are

$$r_+ + r_c + r_0 = 0 \tag{36}$$

$$r_+ r_c r_0 = -\frac{6}{\lambda} M \tag{37}$$

$$r_+ r_c + r_c r_0 + r_0 r_+ = -\frac{3}{\lambda} \tag{38}$$

By expanding the denominator of Eq. (35), we get

$$[r_+^3 - (r_c + r_0)r_+^2 + r_+ r_c r_0]^2 \tag{39}$$

Putting Eqs. (36) and (37) into (39), we find that Eq. (40) becomes

$$4\left(r_+^3 - \frac{3}{\lambda} M\right)^2 \tag{40}$$

Substituting Eq. (40) into Eq. (39) gives the black hole the entropy,

$$S_+ = \frac{7}{2} \frac{1}{4} A_+ \tag{41}$$

Since the entropy of the Klein–Gordon field is one fourth the area of the event horizon, the the entropy of the Dirac field is $3\frac{1}{2}$ times that of the Klein–Gordon field.

5. DISCUSSION

We take the black hole event horizon and the cosmological event horizon as two independent thermodynamic systems and calculate the entropy of the Dirac field via the membrane model. The entropy is proportional to the total area. This is consistent with previous results. ones. Hence the idea of that black hole entropy comes from the vicinity of the event horizon has some plausibility.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China (19773003).

REFERENCES

1. Gao Changjun and Zhao Zheng (2000). Membrane model: An improvement of the brick wall model, preprint.
2. G't Hooft (1985). *Nucl. Phys. B* **256** 727.
3. J. D. Bekenstein (1973). *Phys. Rev. D* **7** 2333.
4. J. Ho, W. T. Kim, Y. J. Park, and H. Shin (1997). *Class. Quantum Grav.* **14** 2617.
5. J. M. Bardeen, B. Carter, and S. W. Hawking (1973). *Commun. Math. Phys.* **31** 161.
6. Liu Wenbiao and Zhao Zheng (2000). *Phys. Rev. D* **61** 063003, U68.
7. S. P. de Alwis and N. Ohta (1995). *Phys. Rev. D* **52** 3529.
8. S. W. Hawking (1975). *Commun. Math. Phys.* **43** 199.
9. Zhao Zheng, *The Thermal Character of Black Holes and the Singularity of Space-time* (Beijing Normal University Press, Beijing, 1999).
10. Zhijian Luo and Jianyang Zhu (1999). *Acta Physica Sinica* **48** 395.